Axial Laminar Flow of a Non-Newtonian Fluid in an Annulus

DONALD W. McEACHERN
University of Wisconsin, Madison, Wisconsin

The equation of motion has been solved for steady axial, laminar, isothermal flow of an Ellis model fluid in a conduit of annular cross section. Tables are presented which may be used to obtain flow curves for annular flow of fluids whose Ellis parameters are known.

The Ellis fluid predictions are compared to experimental data on dilute polymer solutions flowing in annuli. Ellis fluid predictions of flow curves are also compared to those predicted by the generalized Newtonian fluid and the power law fluid.

This paper shows how viscometric data may be used to predict the flow rate–pressure drop relationship for steady, isothermal, axial flow of an incompressible non-Newtonian fluid in a long conduit of annular cross section. The constitutive equations used are for the generalized Newtonian fluid and one special case of this fluid, the Ellis fluid model.

The solution of the z component of the equation of motion for an Ellis fluid is given. Tables are presented which enable the pressure drop–flow rate relationship for the annulus to be predicted for fluids whose Ellis parameters are known. The relationship of flow rate and pressure drop (flow curve) for annular flow is given in integral form for the generalized Newtonian fluid, and the evaluation of this integral is discussed.

The results of the calculations for the generalized Newtonian fluid and Ellis fluid are compared with experimental data for dilute polymer solutions flowing in annuli and with the results of the analytical solution of the annulus problem by Fredrickson and Bird (*1*), who used the power law model.

Annular flow problems are studied because solutions are applicable to flow in heat exchange equipment, extruders for molten plastics, and flow of drilling mud in oil well drilling. Analytic solutions for axial annular flow have been given by Slibar and Paslay (*2*), Laird (*3*), and Fredrickson and Bird (*1*) for the Bingham fluid. Approximate solutions to this problem have been given by Volarovich and Gutkin (*4*) and van Olphen (*5*) for the Bingham fluid. Fredrickson and Bird (*1*) have also given an analytic solution of the annulus problem using the power law fluid. The methods of solution used here for the Ellis fluid are quite similar to those used by Fredrickson and Bird (*1*) for the power law fluid. The need for a more complex fluid model to be applied to the annulus problem was pointed out by the failure of the power law results to describe the annular flow data of Fredrickson (*14*).

FLOW CONDITIONS

Consider the laminar isothermal flow of an incompressible fluid in a conduit of annular cross section where the energy dissipation by viscous forces is neglected. The velocity is assumed to be: $v_r = 0$, $v_\theta = 0$, $v_z = v_z(r)$, with values on the outer wall ($r = R$) and inner wall ($r = kR$) given as $v_z(R) = v_z(kR) = 0$.

CONSTITUTIVE EQUATIONS

The constitutive equations used will be for the incompressible generalized Newtonian fluid† and for a special case of this fluid, the Ellis fluid. For the form of the velocity assumed above, the generalized Newtonian (*6, 7*) fluid is

$$\tau_{rz} = -\eta \frac{dv_z}{dr} \tag{1}$$

where η is the generalized Newtonian viscosity and for isothermal annular flow is a function of the second invariant of the shear stress tensor (*8*). The quantity τ_{rz} is a component of the shear stress tensor. The Ellis fluid for the assumed velocity form is

$$-\frac{dv_z}{dr} = \frac{1}{\eta_o}\left[1 + \left|\frac{\tau_{rz}}{\tau_{1/2}}\right|^{\alpha-1}\right]\tau_{rz} \tag{2}$$

where η_o, $\tau_{1/2}$, and α are numerical constants which may be determined easily from viscometric data. From Equation (2), the Ellis fluid may be seen to be composed of a Newtonian term plus a power law term. The Ellis fluid exhibits a constant value of viscosity at low shear stresses, and at higher shear stress the plot of log η vs. log $\sqrt{\tau^2_{rz}}$ becomes a straight line with slope $1 - \alpha$. When $\alpha > 1$ the viscosity η, represented by the Ellis fluid, approaches zero for very large shear stresses rather than a constant value as many real fluids are observed by experiment to do. An extensive review of the Ellis fluid has been given by Matsuhisa and Bird (*13*).

SOLUTION OF THE EQUATION OF MOTION

Ellis Fluid

To predict the flow curves for an annulus, only the z component of the equation of motion must be considered.

† The generalized Newtonian fluid model does not indicate that there are any normal stresses present in the flow being considered. Coleman and Noll (*9*) have shown for the simple fluid of Noll (*10*) and for the Rivlin-Ericksen fluid (*11*) that for steady, laminar flow of these incompressible fluids in the absence of end effects that the pressure drop-flow rate relationship depends only on one rheological property of the fluid which may be defined in terms of the generalized Newtonian viscosity η. So any normal stresses which may be present do not effect the pressure drop-flow rate relationship. Therefore, flow curves predicted by the generalized Newtonian fluid are equivalent to those predicted by the simple fluid and the Rivlin-Ericksen fluid.

Donald W. McEachern is with Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

The velocity form assumed above, coupled with the Ellis fluid model [Equation (2)], leads to the following form for the z component of the equation of motion:

$$\frac{1}{r}\frac{d}{dr}(r\tau_{rz}) = P \quad (3)$$

where P is defined as

$$-\frac{dp}{dz} + \rho g_z$$

It is convenient to introduce the dimensionless quantities: dimensionless velocity $\phi = v_z\,\eta_o/\tau_{1/2}\,R$; dimensionless radial distance $\xi = r/R$; dimensionless shear stress $T = \tau_{rz}/\tau_{1/2}$; dimensionless pressure gradient $P^* = PR/2\tau_{1/2}$.

The first integral of Equation (3) gives the shear stress

$$T = P^*\left(\xi - \frac{\lambda^2}{\xi}\right) \quad (4)$$

Here the integration constant λ has the special meaning $T = 0$ at $\xi = \lambda$. Since the shear stress changes sign in the annular space $1 \geqq \xi \geqq k$, it is necessary to consider separately the two regions $1 \geqq \xi \geqq \lambda$, and $\lambda \geqq \xi \geqq k$.

By substitution of Equation (2) in Equation (4) and by the use of the boundary conditions, the velocity profile is given upon integration.

$$\phi_+ = P^* \int_\xi^1 \left(\xi - \frac{\lambda^2}{\xi}\right) d\xi + (P^*)^\alpha \int_\xi^1 \left(\xi - \frac{\lambda^2}{\xi}\right)^\alpha d\xi \quad 1 \geqq \xi \geqq \lambda \quad (5)$$

and

$$\phi_- = P^* \int_k^\xi \left(\frac{\lambda^2}{\xi} - \xi\right) d\xi + (P^*)^\alpha \int_k^\xi \left(\frac{\lambda^2}{\xi} - \xi\right)^\alpha d\xi \quad \lambda \geqq \xi \geqq k \quad (6)$$

The velocity at $\xi = \lambda$ given by the Equations (5) and (6) must be equal, so an equation determining λ may be obtained by setting $\phi_+(\lambda) = \phi_-(\lambda)$. The value of λ for integer values of α may be evaluated by expressing the integrands of Equations (5) and (6) by means of binomial expansions. When the order of integration and summation are interchanged and the indicated integrations are performed, the resulting algebraic equations for λ is

$$\frac{\frac{1}{2}(1-k^2) - \lambda^2 \ln(1/k)}{P^{*\alpha-1}} = \sum_{\substack{i=0 \\ i \neq \frac{\alpha+1}{2}}}^{\alpha} L_i \lambda^{2i} + M\lambda^{\alpha+1} \quad (7)$$

where

$$L_i = (-1)^{i+1}\binom{\alpha}{i}\frac{[1 + (-1)^\alpha k^{\alpha-2i+1}]}{\alpha - 2i + 1}$$

$$M = \binom{\alpha}{\frac{\alpha-1}{2}}(-1)^{\frac{\alpha-1}{2}} \ln(1/k) \qquad \alpha \text{ odd}$$

$$M = 2\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\frac{1}{\alpha - 2i + 1} \qquad \alpha \text{ even}$$

In the case of the Newtonian fluid $\alpha = 1$, Equation (7) gives the well-known value of

$$\lambda = \sqrt{1/2\,(1-k^2)/\ln(1/k)}$$

The right-hand side of Equation (7) is precisely the same form as the determining equation for λ obtained by Fredrickson and Bird (*1*) for the power law fluid† where in Equation (7) α takes the role of the power law s.

If the integral of the velocity over the cross section is performed by expressing the integrand by binomial expansion, the flow rate is given by

$$Q = \frac{\tau_{1/2}\pi R^3}{\eta_o}\left\{P^*\left[\lambda^4 \ln(1/k) - \lambda^2(1-k^2) + 1/4\,(1-k^4)\right] + P^{*\alpha}\left(\sum_{\substack{i=0 \\ i \neq \frac{\alpha+3}{2}}}^{\alpha+1} E_i\lambda^{2i} + F\lambda^{\alpha+3}\right)\right\} \quad (8)$$

where

$$E_i = \binom{\alpha+1}{i}(-1)^i\left[\frac{1 + (-1)^\alpha k^{\alpha+3-2i}}{\alpha + 3 - 2i}\right]$$

$$F = \binom{\alpha+1}{\frac{\alpha-1}{2}}(-1)^{\frac{\alpha-1}{2}} \ln(1/k) \qquad \alpha \text{ odd}$$

$$F = 2\sum_{i=0}^{\alpha+1}\binom{\alpha+1}{i}(-1)^i\frac{1}{2i - \alpha + 1} \qquad \alpha \text{ even}$$

The first term of the expression for volume flow rate given above is the same form as the expression for annular flow of a Newtonian fluid. The second term is just the same as the expression given by Fredrickson and Bird for the power law fluid where α takes the role of the power law parameter s and $(\eta_o^{1/\alpha})\,(\tau_{1/2}^{(\alpha-1)/\alpha})$ takes the role of the power law parameter m.

The limiting values of Q evaluated from Equation (8) are seen to be:

Newtonian flow in a circular tube $\alpha = 1$, $k = 0$, $\mu = \eta_o/2$:

$$Q = \frac{\pi R^4 P}{8\mu} \quad (9)$$

Flow of an Ellis fluid in a tube $\alpha \neq 1$; $k = 0$:

$$Q = \frac{\pi R^3}{4\eta_o}\left(\frac{PR}{2}\right)\left[1 + \frac{4}{\alpha+3}\left(\frac{PR}{2\tau_{1/2}}\right)^{\alpha-1}\right] \quad (10)$$

Flow of a Newtonian fluid in an annulus $\alpha = 1$, $\mu = \eta_o/2$:

$$Q = \frac{\pi R^4 P}{8\mu}\left(1 - k^4 - \frac{(1-k^2)^2}{\ln(1/k)}\right) \quad (11)$$

Flow of an Ellis fluid in a narrow annulus $\alpha \neq 1$, $k \approx 1$.

Matsuhisa and Bird (*13*) have given an expression for flow of an Ellis fluid in narrow annuli. Their expression was obtained by multiplying the Ellis parallel-plate solution by a Newtonian curvature correction.

$$Q = \frac{\pi R^3}{3\eta_o}\left(\frac{PR}{2}\right)\epsilon^3\left(1 - \frac{1}{2}\epsilon + \frac{1}{60}\epsilon^2 + \ldots\right)\left[1 + \frac{3}{\alpha+2}\left(\frac{PR\epsilon}{\tau_{1/2}2}\right)^{\alpha-1}\right] \quad (12)$$

Matsuhisa and Bird recommended this expression for annuli where $k > 0.8$. However, the results obtained here show that Equation (12) is accurate to $\pm 5\%$ for $k \geqq 0.3$ and $\alpha \leqq 4.0$.

The values of λ and $Q^* = Q\eta_o/\tau_{1/2}\pi R^3$ have been calculated from Equations (7) and (8), respectively, for the following ranges of the parameters:

† The power law fluid is expressed for axial annular flow as

$$\tau_{rz} = -m\left|\frac{dv_z}{dr}\right|^{n-1}\frac{dv_z}{dr}$$

The quantity s is defined as $1/n$.

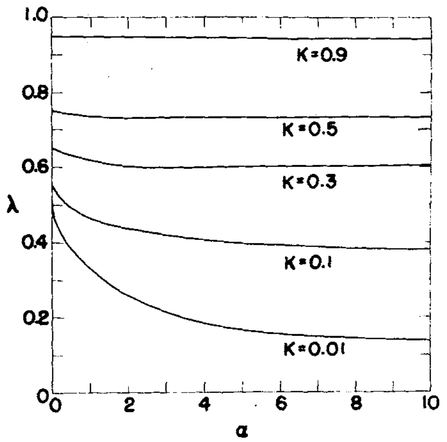

Fig. 1. λ vs. Ellis parameter α for $PR/2\tau_{1/2} = 1$.

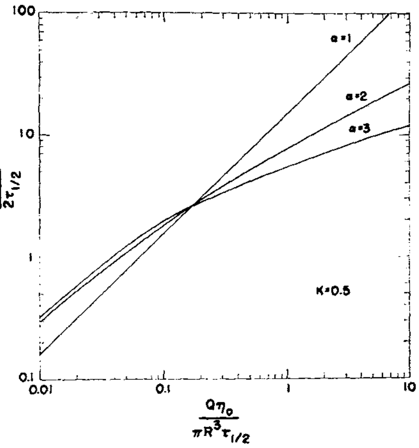

Fig. 3. $Q\eta_0/\pi R^3\tau_{1/2}$ vs. $PR/2\tau_{1/2}$ for $k = 0.5$ calculated from Equation (8).

$P^* = 0.1, 0.2, 0.5, 1.0 \ldots 1000$
$\alpha = 1.0, 1.5\dagger, 1.75\dagger, 2.0, 2.25\dagger, 2.50\dagger, 2.75\dagger, 3.0, 4.0 \ldots 6.0$
$k = 0.01, 0.1, 0.2 \ldots 0.9$

These results are presented in tabular form in the Appendix††. Also shown in the Appendix are some example problems demonstrating the use of these tables. Some of the tabular material is plotted in Figures 1 to 3. These graphs serve to demonstrate the effects of the geometry and fluid parameters on the value of λ and the pressure drop–flow rate relationship. All possible combinations of the values of P^*, α, and k are not included in the tables in the Appendix, because the calculation procedure breaks down for large k and large α.

Generalized Newtonian Fluid

The solution to the annulus problem under consideration here has been given for the generalized Newtonian fluid by Fredrickson (*14*) and by Coleman and Noll (*9*). The volume flow rate is

$$Q = \frac{\pi R^4 P}{2} \int_k^1 \frac{1}{\eta}\left(\xi - \frac{\lambda^2}{\xi}\right)(\xi^2 - k^2)\, d\xi \qquad (13)$$

when

$$\lambda^2 = \int_k^1 (\xi/\eta)\, d\xi \Big/ \int_k^1 \frac{d\xi}{\xi\eta} \qquad (14)$$

To compare results of the Ellis fluid solution with the more generally valid results of the generalized Newtonian fluid, calculations were carried out using Equations (13) and (14) to predict flow curves from available experimental data.

The calculation of flow rate involves a trial and error procedure, because η, which appears in the integral of Equation (14), is implicitly dependent on λ. The required calculation may be carried out graphically or by numerical techniques. In the investigation reported here, these calculations were carried out numerically on a computing machine.

Experimental viscometric data (η vs. τ_{rz}) were entered into the machine as a table, and linear interpolation was used to find values of η between table entries. The integrations in Equations (13) and (14) were carried out by using Simpson's rule one-hundred times over the cross section.

† Calculations for these values were carried out by using Equations (13) and (14) with the Ellis model definition of η.

†† Tabular material has been deposited as document 8677 with the American Document Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for $1.75 for photoprints or $2.50 for 35-mm. microfilm.

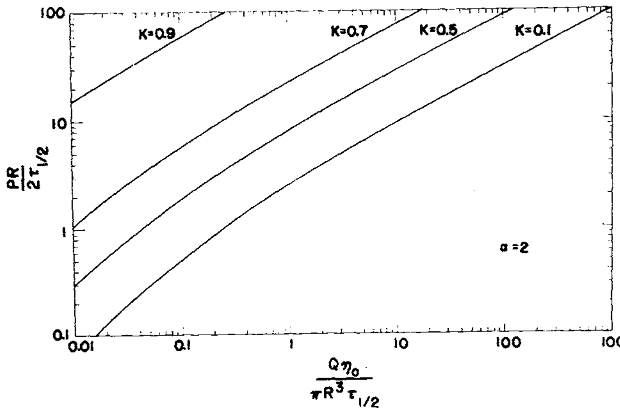

Fig. 2. $Q\eta_0/\pi R^3\tau_{1/2}$ vs. $PR/2\tau_{1/2}$ for $\alpha = 2$ calculated from Equation (8).

EXPERIMENTAL APPARATUS

Viscometric Apparatus

Viscometric data were gathered for eight aqueous solutions of a hydroxyethylcellulose polymer, Natrosol, with a capillary tube viscometer. The instrument description and technique for analysis of the data are given in reference *12*. The accuracy of these viscometric data is estimated to be about ± 5%. The Ellis model parameters determined from these viscometric data are given in Table 1.† Also shown in Table 1 are the Ellis model parameters evaluated from the viscometric data of Frederickson (*14*). In Table 2† the power law parameters for the fluids used in this investigation and for Frederickson's fluids are given.

Annulus Apparatus

The annulus data were gathered on an ordinary flow loop apparatus. Annuli with radius ratios k of 0.250 and 0.504 were used. The outside tube of both annuli was 12-ft. long, smooth brass tube with an outside diameter of 0.500 in. The cores were formed by stainless steel tubes with diameters of 0.125 and 0.252 in. Flow rates were measured by weighing fluid collected during a measured time interval. Pressure drops were measured by water-over-mercury and water-over-oil manometers connected to various combinations of 1/16-in.-diameter pressure taps drilled through the outside tube 100, 150, 175, and 200 outside tube diameters from the test section inlet.

The 0.125-in.-diameter core was centered by two supports, one placed at the inlet of the test section and the other placed

† See second footnote on this page.

TABLE 3. SUMMARY OF NON-NEWTONIAN ANNULAR FLOW DATA AND THEORETICAL PREDICTIONS

Average absolute and maximum errors in % with which PR/2 is predicted by:

Fluid	Generalized Newtonian fluid		Ellis fluid		Power law fluid	
	Average	Maximum	Average	Maximum	Average	Maximum
Annulus $k = 0.504$ (77°F.)						
0.7% Natrosol-H	1.9	5.7	3.1	6.9	8.7	29.3
0.5% Natrosol-H	3.6	6.7	2.9	4.7	1.6	7.1
0.3% Natrosol-H	1.1	2.1	1.3	3.4	1.7	4.1
1.0% Natrosol-G	3.0	3.9	2.5	4.7	8.0	20.0
Annulus $k = 0.250$ (77°F.)						
0.7% Natrosol-H	2.6	6.6	3.7	8.3	3.6	8.0
0.5% Natrosol-H	3.5	7.3	2.4	4.6	4.1	6.5
0.3% Natrosol-H	3.8	7.6	3.9	6.5	8.1	13.9
1.0% Natrosol-G	6.3	13.3	1.9	5.9	4.6	11.8
Annulus $k = 0.624$ (85°F.) data of Fredrickson (*14*)						
1.0% CMC-70-H	—	—	8.4	19.1	7.6	14.6
3.5% CMC-70-M	—	—	6.8	32.5	63.9	170
2.5% CMC-70-M	—	—	7.7	11.1	24.2	60.5
Annulus $k = 0.363$ (85°F.) data of Fredrickson (*14*)						
3.5% CMC-M	—	—	4.1	10.5	69.4	204
2.5% CMC-M	—	—	4.9	11.9	15.3	54.6

at the outlet of the test section (88 outside pipe diameters downstream from the last pressure tap). The 0.252-in.-diameter core was held by four fixed supports placed 0, 24, 264, and 288 outside pipe diameters from the test section inlet. Tension was held on both cores to lessen sagging of the cores between the supports.

The cores were checked for eccentricity by flowing sucrose solutions in laminar motion through the annuli and by comparing the data with Lamb's solution [Equation (11) of this paper]. These experiments agreed with Equation (11) to within 4.8% for the annulus $k = 0.504$ and to within 0.08% for the annulus with $k = 0.250$.

COMPARISON OF DATA AND THEORY

Table 3 shows the accuracy with which the annular data are described by the generalized Newtonian fluid treatment, the Ellis fluid treatment, and the power law treatment. Some of the experimental data of this investigation are plotted with the generalized Newtonian fluid predictions and the Ellis fluid predictions in Figures 4 and 5. Only one curve is shown for each annulus, because the generalized Newtonian and Ellis predictions are almost indistinguishable when plotted. In Figure 6, the flow curves predicted by using the Ellis fluid and the power law fluid are shown plotted with the data of Fredrickson (*14*).

All the experimental annulus flow curves are predicted well within the accuracy of the experiment by the generalized Newtonian and Ellis viscometric representations. This agreement emphasizes that the assumptions made during the theoretical treatments of the annulus problem were indeed accurate approximations of the experimental conditions. The matching of the range of shear stresses in the capillary tube experiment with the shear stress at the outside wall of the annulus contributed significantly to the accuracy of the annulus predictions. This is demonstrated in Figure 6. Two lines which were obtained from the power law results of Fredrickson and Bird are shown in Figure 6, plotted with the experimental annulus data of Fredrickson. The equal length dashed line corresponds to values of m and n evaluated from the linear portion of the $\log \eta$ vs. $\log \tau_{rz}$ curve. These power law parameters describe the viscometric data between shear stresses of 3,500

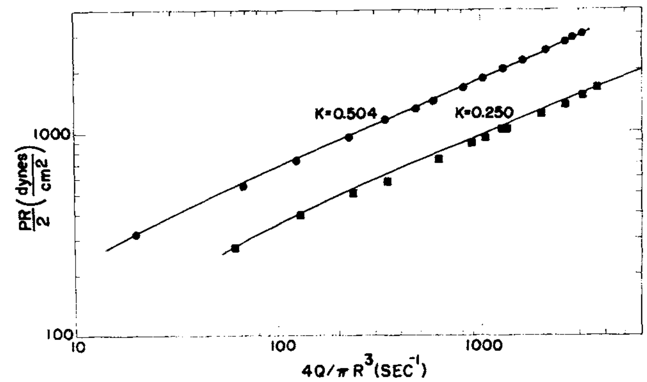

Fig. 4. Flow curves for 0.7% Natrosol-H solution flowing in annuli (77°F.). Solid lines are predictions of generalized Newtonian and Ellis fluid treatments.

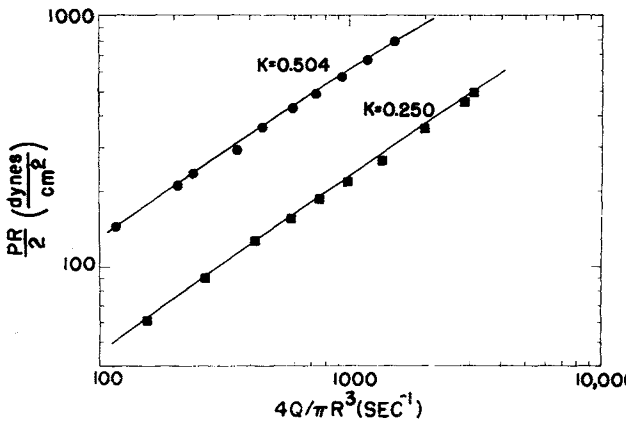

Fig. 5. Flow curves for 0.3% Natrosol-H solutions flowing in annuli (77°F.). Solid lines are predictions of generalized Newtonian and Ellis fluid treatments.

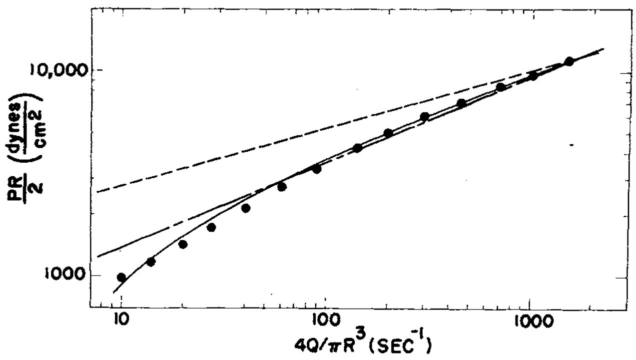

Fig. 6. Flow curves for 3.5% CMC-70-M solution flowing in an annulus with $k = 0.363$ (85°F.) (*14*). Solid line is the prediction of Ellis fluid. Dashed line (equal length dashes) is prediction of power law for parameters evaluated over linear portion of log η vs. log τ_{rz} curve. Dashed line (unequal length dashes) represents power law prediction for parameters evaluated for shear stress 1,000 to 7,000 dynes/sq.cm.

to 11,500 dynes/sq. cm. The unequal length dashed line was obtained from values of m and n evaluated from the viscometric data in the range of shear stress 1,000 to 7,000 dynes/sq. cm. The latter line describes the annulus flow curves with an average absolute error of only 13.4%, whereas the former is in error 69%. The experimental values of shear stress on the outside wall of the annulus for the data plotted in Figure 6 are between 600 and 7,000 dynes/sq. cm. This, as well as the rest of the annulus data, indicates that the shear stress at the outside wall of the annulus of interest might suggest the upper limit of the range of shear stresses in which viscometric measurements need to be made or fluid parameters should be evaluated if annulus flow curves are to be predicted.

CONCLUSION

The following recommendations for the estimation of annulus flow curves are offered.

If annular flow curves are to be predicted for non-Newtonian fluids, it is desirable to have viscometric data which extend from as low a shear stress as possible to a value of shear stress as high as those expected on the outside wall of the annulus. If these data can be described with sufficient accuracy by the Ellis model, then the Ellis model parameters should be evaluated. These parameters should be used in referring to the tables given in the Appendix where the flow curves can be evaluated by simple arithmetic.

If the viscometric data are not described by the Ellis model with sufficient accuracy, Equations (13) and (14) may be used to estimate the flow curves. The evaluation of the integrals can be carried out as suggested above or by some other numerical or graphical procedure.

The data of this investigation demonstrate that the power law viscometric representation can be used with the solution of the annulus problem given by Fredrickson and Bird (*1*) to estimate flow curves for the annulus. However, when using the power law results, particular attention must be paid to the range of shear stress over which the power law parameters are evaluated.

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NOTATION

E_i see Equation (8)
F see Equation (8)
g_z = component of gravitational force in z direction
k = radius ratio, ratio of radius of annulus core to inside radius of outer annulus tube, $(0 \leq k \leq 1)$
L_i = see Equation (7)
M = see Equation (7)
m = power law parameter
n = power law parameter
P = combined pressure and gravity force, $P = -dp/dz + \rho g_z$
P^* = dimensionless pressure gradient, $P^* = PR/2\tau_{1/2}$
p = pressure
Q = volumetric flow rate
Q^* = dimensionless flow rate, $Q\eta_o/\tau_{1/2}\pi R^3$
R = inside radius of outside tube of annulus
r = radial coordinate
s = power law parameter, $s = 1/n$
T = dimensionless shear stress
v_i = component of velocity in i^{th} direction
z = axial coordinate

Greek Letters

α = Ellis parameter
η = viscosity defined by Equation (1)
η_o = Ellis parameter
θ = angular coordinate
λ = value of r/R where the shear stress in the annulus is zero
μ = Newtonian viscosity
ξ = dimensionless radial distance
ρ = fluid density
$\tau_{1/2}$ = Ellis parameter
ϕ = dimensionless axial velocity
ϵ = $1 - k$

Subscripts

$+$ = range $\lambda \leq \xi \leq 1$
$-$ = range $k \leq \xi \leq \lambda$

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